Application of Wavelets to the Analysis of Multiscale Structures

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Abstract. We demonstrate capabilities of wavelet transform on the measured magnetic field of a collisionless shock. We show that application of the wavelet transform using the first derivative of Gaussian ($G_1$) wavelet to the total magnetic field of the shock in the ramp vicinity allows to identify reliably the position and the width (gradient scale) of the ramp transition. We apply the Morlet transform to the analysis of the downstream wave properties. Using vector wavelet transform we develop a local version of the wave polarization analysis.

1. Introduction

The standard method of analysis of a time series is Fourier transform, which has a number of drawbacks. For example, it cannot be applied to wavepackets which are only several wavelengths long. Neither can it separate localized structures which contribute into power spectrum independently of their position. Wavelet transform is free of these drawbacks and allows retain information about position and scale (frequency) as well [Jurgé, 1992; Lewalle, 1994]. It is, therefore, an appropriate tool for the analysis of complex systems, which contain a number of different scales and in which stationary localized structures coexist with quasiperiodic time-dependent features [Dudok de Wit et al., 1995]. An example of such system is a collisionless shock profile, in which on a presumably sharp gradient of the magnetic field (ramp) large amplitude waves may be superimposed (see, for example, Scudder et al. [1986]), making identification of the ramp itself and determination of the wave features difficult [Rezeau et al., 1998]. In the present paper we apply wavelet transform to a shock profile (magnetic field) to demonstrate capabilities of the method. The emphasis is on the possibility of extracting useful information from a single spacecraft observations without invoking other plasma parameter measurements. We restrict ourselves only to the magnetic vector data. The paper is organized as follows.

In section 2 we give theoretical background of the wavelet transform and the method presented in the paper. In section 3 we apply the wavelet transform to the ramp transition region in the shock profile for the determination of the ramp scale. In section 4 we apply Morlet transform for the local determination of the downstream wave properties in the same shock profile.

2. Theoretical Background

Any localized function $\psi(t)$, so that $\int_{-\infty}^{\infty} |\psi|^2 dt < \infty$, can be chosen as a mother wavelet, if it satisfies the admissibility criterion $\int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} \psi dt = 0$ [Lewalle, 1994]. If this is the case, a family of the wavelets is built according to the following prescription:

$$\psi[a, d](t) = a^{-1/2} \psi\left(\frac{t-a}{d}\right),$$

where $a$ is the position of the wavelet and $d$ is its scale (time and duration, respectively, in the case of time series).

For any $f(t)$ its wavelet transform $W[f](a, d)$ is defined as follows:

$$W[f](a, d) = \int_{-\infty}^{\infty} f(t) \psi^* [a, d](t) dt.$$

In reality $f(t)$ is not a continuous function but a time series, so that integration in (2) should be substituted by summation. An obvious immediate advantage of the method is that the time series does not have to be stationary, and there is no need to detrend it.

It is clear that the result of the transform depends on the choice of the mother wavelet (in general, arbitrary except the admissibility condition), so that this choice should be dictated by our goals. In what follows we will be interested in the analysis of two different features, which in the ideal case look as a) a sharp jump of the magnetic field, or b) monochromatic wave. It has been shown [Gedalin et al., 1998] that the first derivative of Gaussian ($G_1$) (see Figure 1a)

$$G_1(t) = t \exp(-t^2/2) = -d/dt \exp(-t^2/2)$$

is especially suitable for the identification of the features of the first kind, while waves are best analyzed with the use of the complex Morlet ($\tilde{M}$) wavelet [Dudok de Wit et al., 1995].

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Let us consider now the $M$ transform of a monochromatic wave $f = \cos(\omega t)$. The transform is easily calculated:

$$\mathcal{W}_M(a,d) = \frac{\pi}{2} \exp(-\text{imag}(-a - 2\pi^2/2))$$

with $\text{imag}\left(-a - 2\pi^2/2\right)$. The second term in square brackets is always small and we shall neglect it henceforth. Then the wavelet power spectrum is

$$\mathcal{P}_M(a,f) = \frac{\pi}{2} \exp(-4\pi^2 f_0^2/9 - f^2/2)$$

where we defined $f_0 = \omega/\sqrt{2}$ and $f = \sqrt{2}/d$. It is easy to see that the spectral power has a sharp maximum at $f = f_0$.

In reality wavepackets are always of a finite length, so we examine the $M$ transform of the following wavepacket:

$$f = \exp(-|t - t_0| - (t - t_0)^2/2\sigma^2)$$

The spectral power is easily calculated in the following form:

$$\mathcal{P}_M(a,f) = \frac{d^2 + T^2}{\exp(-2\pi^2 K^2 f_0^2 - f^2)} - (a - \text{imag}(\sigma^2 d^2 + T^2))$$

where, as above, $f_0 = \omega/2\sigma$, $f = \sqrt{2}/d$, and $K^2 = d^2 + T^2$. Again there is a sharp maximum at $f = f_0$ for fixed $\sigma$. We can also estimate the temporal width $\Delta_t$ of the $M$ transform (which is centered on $t = t_0$ as is the initial wavepacket) and uncertainty in the frequency determination $\Delta_f/\Delta_t$ as:

$$\Delta_t = \sqrt{d^2 + T^2} \approx \sqrt{1 + (f_0 T)^2}/f_0$$

$$\Delta_f/\Delta_t \approx \frac{1}{2\pi K} \approx \frac{1}{2\pi} \left(1 + (f_0 T)^2\right)$$

which shows that when $f_0 T > 1$ the temporal width of the $M$ transform equals the width of the wavepacket, and the uncertainty of the frequency determination is $1/2\pi$. Expressions (12)-(13) show that even when $f_0 T = 1$, that is, when the wavepacket is short and contains only $1$ waveperiod, the precision of the determination of its parameters using the $M$ transform is rather good.

As is seen from this analysis, the $M$ transform of a wavepacket looks as a stripe in the $t - f$ plane, which is parallel to $d$-axis. It can be shown that the $M$ transform of the step-like profiles looks as a "tare" diverging to large $d$ and can be easily separated from the wave features.

In what follows we plan to apply $M$ transform to the measurements of the magnetic field vector $B = (B_x, B_y, B_z)$. As can be seen from above, the $M$ transform effectively extracts quasi-monochromatic wave packets, and $W_B = W_B(i\omega) = B(i\omega,a,d)$, where $i = x, y, z$ and the function $q(a,d)$ is the same for all three components. Thus we can treat $W = (W_x, W_y, W_z)$ locally as a magnetic field of a monochromatic wave (apart of insignificant common factor $q$).

Let us consider such a wave, in which $B = B_1 + iB_2$, where $B_1$ and $B_2$ are real. Presence of $i$ means phase shift
of π/2. It can be shown that, for example, the usual def-
inition of the degree of circular polarization \( P_c \) can be re-
formulated in the vector form as follows (cf. \textcite{Rybicki and Lightman} [1979]):

\[
P_c = \frac{2|B_1 \times B_2|}{|B_1| \cdot |B_2|},
\]

and, if \( P_c \neq 0 \), the propagation direction is given by the
following unity vector:

\[
\hat{n} = \frac{B_1 \times B_2}{|B_1| \cdot |B_2|}.
\]

Translating this into the language of M-transform, one finds

\[
P_c = \frac{2|\text{Im} W \times \text{Re} W|}{|\text{Im} W| \cdot |\text{Re} W|},
\]

\[
\hat{n} = \frac{|\text{Im} W \times \text{Re} W|}{|\text{Im} W| \cdot |\text{Re} W|}.
\]

The above circular polarization \( P_c \) coincides with the helic-
ity as defined by \textcite{Ryssza-Vérban et al.} [1994]. It is worth
noting that because of the nature of the wavelet transform \( W \)
always completely (elliptically) polarized and the de-
gree of linear polarization \( P_l = 1 - P_c \).

3. Ramp Identification

Identification of the ramp position and width is one of the
most important problems of the observational shock physics,
since the ramp is believed to be the place where most ener-
getic processes in the shock front occur (ion deacceleration
and reflection, electron heating, dissipation necessary for
shock stationarity), and knowledge of its scale and behav-
ior (stationary or nonstationary, one-dimensional or three-
dimensional) is crucial for understanding shock physics.
Yet this task may be complicated by presence of superim-
posed wave features. Usual methods of averaging and cross-
correlating \textcite{Scudder et al., 1986} may be not quite appropri-
ate since averaging smoothes out the fine scales which are
of utmost interest. Wavelet technique, both direct \textcite{Gedalin et al., 1998} and combined with cross-correlation \textcite{Ryssza et al., 1998} is shown to provide more information about the
ramp scale and position. In this section we demonstrate the
capabilities of the wavelet transform in the above task by
applying the \( G_1 \) transform to the magnetic field profile of
the shock, measured by \textcite{ISEE} I on July 23, 1978 at 0159
UT. The shock is a high Mach number \( M_{shock} = 4.4 \), high \( \beta = 1.14 \), quasi-perpendicular \( \theta_{IM} = 72^\circ \) shock \textcite{Newbury et al., 1998}. The total magnetic field and components are shown in Figure 2. The ramp transition is rather sharp and
quite clear, although substantial amount of wave activity is
superimposed on it and persists well beyond the ramp. The
upstream wave activity is relatively weak and we will not
study the upstream region in the present paper. Time in the
figure is measured in seconds with \( t = 0 \) corresponding to the
first data point of the data piece at 01:49:00 UT. Verti-
cal lines delimit the central part of the shock, where we are
interested in the ramp identification, and the downstream re-


gion, where we will study the wave polarizations features.

Figure 3 shows the wavelet spectral power for \( G_1 \) and \( M \)
transforms applied to the total magnetic field in the central
part of the magnetic field profile. Time (and position) is on
the horizontal axis for all panels. The vertical axis on the
\( G_1 \) panel shows scale (duration) \( s \) in seconds, while for the
\( M \) panel frequency \( (1/s) \) in the logarithmic scale is given on
the vertical axis. The power spectrum is shown by a
grey scale image. The \( G_1 \) transform panel clearly shows the
transition by the clear cut influence cone converging in
the small scale limit to the position of the ramp transition.

The \( G_1 \) transform shows two additional distinct sharp gradi-
ents features downstream which are not exactly step-like fea-
tures as the ramp is supposed to be. It is rather insensitive to
wave activity despite the large amplitude of the waves and
thus separates the supposedly stationary part of the shock
front from the time-dependent features. More reliable con-
clusion about stationarity, however, requires analysis of a
second spacecraft measurements \textcite{Gedalin et al., 1998; New-
bury et al., 1998} which is not done in the present paper.

The \( M \) transform panel shows presence of a strong down-
stream wave activity in two frequency ranges (see below).
Figure 4 shows the \( G_1 \) transform of the ramp vicinity in
more detail. The ramp transition is identified by the typ-

cal conic shape of the transform rather precisely, despite
significant level of the fluctuations in the ramp itself. The
ramp position can be quite clearly established at \( t = 620 \ s \).
and the ramp width is about 2 s. The second sharp feature ("sub-front") in the downstream is also identified quite clearly although it is not exactly a step-like feature. Its scale is comparable to the scale of the ramp. Ramp identification has been considered in detail by Gedalin et al. 1998 and we restrict ourselves here with the above brief analyses.

4. Downstream Waves

Wavelet technique has been used for identification of wave modes by cross-correlating two spacecraft measurements and applying theoretical dispersion relations [Diodok de Wit et al., 1995]. Such approach has already proved to be successful in the case of whistlers which are not very sensitive to the plasma β and whose dispersion relation is rather model (kinetic of hydrodynamic) independent. Analysis of polarization properties of low-frequency waves allows in principle to make certain conclusions about the nature of these waves [Gray, 1986; Krauss-Varban et al., 1994], even when only single spacecraft measurements are available. Wavelet technique allows to perform this analysis locally without filtering or averaging, thus properly treating even short wave packets. In this section we study downstream waves in the same shock front. Although the amplitude of these waves ~ 0.25% is not very small, we shall treat these waves as almost linear waves, following the principles described in section 2. For each component of the magnetic field \( B_x \) we calculate the corresponding M transform \( W(f, a) \), where \( f = 1/t \) is the wave frequency. The corresponding spectral power matrix is defined as \( P_m(\omega, f) = |W(\omega, a)|^2 \). The total power spectrum is defined as \( P_T = P_{B_x} + P_{B_y} + P_{B_z} \) and serves as a quantitative measure of the wave activity. The power spectra for all three components of the magnetic field are shown in Figure 5. The presentation is as above: time (position) is on the horizontal axis, frequency in the logarithmic scale is on the vertical axis. Here and hereafter in all figures the wavelet transform pattern is masked by cutting out (zeroing) that part of it for which the total power spectrum is low, \( P_T < 0.2 \max(\{} \). This is done to make the representation more clear. It is seen that the wave activity occurs along two distinct stripes. Unfortunately, the frequency of the upper stripe is near the spacecraft rotation frequency 1/3 s (shown by solid line), so that part of wave activity may be of artificial nature. We shall, therefore, concentrate on the lower frequency stripe. It is worthwhile to mention that the typical wave period is \( \sim 5 \) s, while the typical length of the wave packet (as can be seen from Figure 5) is about 40–50 s, that is, only 3–4 wave periods. It is difficult to expect that Fourier analysis can properly treat such waves.

The usual Fourier technique is to filter the data (time series) to retain only frequencies of interest and average over time substantially longer than the wave period [Lazarian et al., 1990]. Averaging does not make sense for wavelet analysis, which is used to extract local information. Moreover, wavelet analysis is capable to differentiate locally between wave packets with different frequencies, even when these wave packets are short and contain only 2–3 wavepe-
riods. The following localized version of the usual wave polarization analysis is based on the analogy between the wave vector $\mathbf{W}(\mathbf{b}, \mathbf{f})$ and the magnetic field of a wave with the frequency $f$, if it could be singled out from the rest of the data. Figure 6 shows one of the polarization characteristics of the waves, which is the ratio of maximum-to-minimum magnetic field, denoted as follows:

$$ R = \frac{1 + \sqrt{1 + f^2}}{1 - \sqrt{1 + f^2}}. $$

(18)

$$ l = \frac{1}{2} (\text{Re} \mathbf{W})^2 + (\text{Im} \mathbf{W})^2, $$

(19)

$$ I_1 = (\text{Re} \mathbf{W})^2 - (\text{Im} \mathbf{W})^2, $$

(20)

$$ I_2 = 2 (\text{Re} \mathbf{W}) \cdot (\text{Im} \mathbf{W}), $$

(21)

which is a direct generalization of the usual definition see, e.g., Rishide and Lifshitz (1979)). Since the $\mathbf{W}$ is completely elliptically polarized, this quantity is the ratio of the major-to-minor axis of the polarization ellipse (magnetic polarization). The wave vector transform panel shows that this ratio is typically greater than unity, with lower values of $R \approx 1.3 - 2$ near the maxima of the power spectrum $P$. Higher values at the edges of the $P$ pattern are of much less physical sense.

As was shown above it is possible to determine the local propagation direction of a wave packet using the complex wavelet transform vector. Figure 7 presents the angle between the wave packet propagation direction found using (17) (bottom panel) and the direction of the magnetic field vector with fluctuations filtered out using the Daubechies-10 wavelet transform and removing 7 finest levels (the magnitude of this filtered magnetic field is shown in the top panel). The middle panel shows the total wavelet power spectrum for reference. Again the attention should be paid to the regions of local (on $f$) maxima of the power spectrum while the edges of the wavelet transform pattern, where the power spectrum drops rapidly, are of much less importance. It should be noted that the determination of the angle $\beta = \arccos \langle \mathbf{n}, \mathbf{b} \rangle$ (where $\mathbf{b}$ is $\mathbf{W}/|\mathbf{W}|$) is reliable only when the circular polarization is substantial, that is, when $R$ is not too large, which is correct in our case. It is clearly seen that the waves typically propagate at small angles $\theta \lesssim 20^\circ$ with respect to the background magnetic field direction. The propagation direction is almost along the magnetic field near the local maxima of the power spectrum. Taking into account the low downstream plasma velocity and large angle (almost $\theta_{B_{0}, V} \approx 50^\circ$) between the downstream magnetic field and the shock normal, one concludes that the corresponding Doppler shift $\omega_{D} / \omega = (V_{||} / V_{\perp}) / \cos \theta_{B_{0}, V} \approx 0.1$ is negligible.

Finally, Figure 8 shows the circular polarization (helicity) $P_{H}$ and general ellipticity $\varepsilon$ of the waves. The last one is defined as follows (cf. Krauss, Lifshitz et al. 1994):

$$ \varepsilon = \text{Re} \left( \frac{W_{\perp}}{W_{\parallel}}, W_{\perp} \right), $$

(22)

where $W_{\perp}$ is the component of $\mathbf{W}$ perpendicular to the $B_{0}$, in plane, while $W_{\parallel}$ is the component in the plane.
Large values of the helicity and substantial ellipticity indicate substantial compressive (magnetoacoustic) component of the wave field [Krauss-Nirih et al., 1994], although more comprehensive analysis of correlations with density and velocity would be necessary for more definite conclusions. It is worthwhile to mention that the plotted polarization characteristics \( P_{r,\theta} \) and \( P_{\phi} \) and the angle \( \theta \) between the wave packet propagation direction and the direction of the ambient magnetic field are not completely independent (some of them are inter-related). Because of the finite wave packet length and numerical uncertainties (finite width of the wavelet transform in the frequency space), the relation may not be as straightforward as one could expect from the finite theory of monochromatic waves and it certainly broken sense to analyze all possible parameters.

5. Conclusions

We demonstrated capabilities of local analysis of a complex multiscale system using wavelets. We have shown that \( G_1 \) wavelet can be used reliably to identify positions and determine widths of shock-Mc features. Applying \( G_1 \) transform to a shock profile we were able to determine the ramp position and gradient scale with a rather high precision, despite noticeable wave activity superimposed on the ramp.

We have also proposed a method for local determination of wave polarization and direction of its propagation, which is promising even for short wave packets. The clear advantage of the method is that it does not require spatial (temporal) averaging, necessary for Fourier analysis, thus allowing obtaining high temporal resolution along with a rather good frequency resolution, even for short wave packets, containing only few wave periods. Application of this method to a high Mach number shock front allowed us to make a preliminary conclusion about predominantily magnetoacoustic nature of the downstream waves.

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References


