On large amplitude MHD waves in high-beta plasma

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Abstract. Downstream of quasi-parallel shocks, the magnetosheath is often observed as a region of high plasma beta, of the order of 10, with large magnetic field fluctuations and relatively weak thermal pressure fluctuations. Since the wave steepening results from the nonlinearity in the plasma thermal properties, these nonlinear magnetic fluctuations that are accompanied by linear pressure fluctuations in a high-beta plasma do not necessarily lead to wave steepening. We develop a nonlinear theory based on MHD and observational evidence to describe such a system. We first perturb the MHD equations assuming the reciprocal of beta and the perturbed pressure to be small parameters and retaining nonlinear terms for the magnetic and velocity perturbations. We then Fourier expand the perturbations and retain terms up to the first order. A nonpropagating mode is present that has the characteristics of a series of tangential discontinuities. A purely incompressible mode is present that propagates obliquely to the background field and is nondispersive and independent of the plasma beta. The field perturbations do not need to be coplanar with the background field and the wavevector. A nonlinear compressible mode is also found. In the limit of small field perturbations, the mode is similar to the linear slow mode in high-beta plasma. The first harmonic field perturbation is coplanar with the background field and the wavevector. When the mode propagates along the field, the density and field strength fluctuations are small at the primary frequency of the wave and are significant at a frequency twice the primary frequency. This frequency doubling phenomenon is most severe for parallel propagation. The compressibility of the mode makes it possible to couple the energy across the magnetopause into the magnetosphere. We show observational evidence for the existence of such modes.

1. Introduction

Large-amplitude fluctuations in magnetized plasmas are often observed in the magnetosheath (e.g., Reiff et al., 1991; Lui et al., 1993) and in the solar wind (e.g., Burlaga, 1971; Belcher et al., 1969; Matthaeus and Goldstein, 1982; Roberts et al., 1987, 1990) and may also be important in solar and astrophysical processes. As their amplitude is large, they carry not only information about the physical processes but also significant energy. They are no longer just passive messengers because their large amplitude is sufficient to alter the medium in which they propagate. In the past, there has been a tendency to ignore the irregular “turbulence” of the magnetosheath and its role on the medium. However, since the waves are strong and potentially important, we must examine how the turbulence in these processes could be different and what role turbulence plays in each process. We believe that in general the major role that these fluctuations play is to convert energy among different types, such as field energy, thermal energy and flow energy. The partition of these different types of energy in the fluctuations indicates their function. Different partitions are associated with different modes. Even fluctuations that at first glance might appear to be noise can be of a particular mode. In a high-$\beta$ plasma, the sound speed and the Alfvén speed are so different that the differences in assessment of the mode will lead to a very different estimate of the response time of a process.

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Physically, the magnetosheath is the region in which the solar wind interacts with the magnetosphere in the form of waves. In the absence of active magnetic reconnection, the magnetopause can shield most wave power from being transmitted into the magnetosphere unless the waves carry significant pressure fluctuations which can oscillate the magnetopause boundary. Therefore among different modes, the compressible modes are particularly important in the solar-terrestrial energy coupling and will be emphasized in our study. Downstream of the quasi-parallel shock, the coupling between the magnetosheath and magnetosphere is greatest [Luhmann et al., 1986; Engelbrecht et al., 1991; Ou et al., 1994], indicating some wave power being transmitted. However, the progress in understanding what are the fluctuations downstream of quasi-parallel shocks and how the waves can be coupled to the magnetosphere has been slow. This study has been motivated to analyze and understand the nature and importance of these fluctuations. In order to identify the modes, and hence the function, of the fluctuations in the magnetosheath, Song et al. [1994] introduced a scheme based on MHD linear theory. We will investigate how nonlinear effects will affect the data analysis.

Unfortunately, a universal theory of the behavior of large-amplitude fluctuations in a broad parameter regime, characterized by the plasma β and frequency relative to the characteristic frequencies of the plasma, has not been established. For example, the cold plasma approach widely used in magnetospheric physics may be useful for low-β plasmas, while the gasdynamic approach is often used for high-β plasmas. A common perception is that the waves, in particular the compressional waves, in high-β plasmas are not far from those in unmagnetized plasmas, sound waves. As we will see in this paper, this perception may not be correct in the nonlinear stage.

In this paper, we develop an analytical MHD approach based on observational evidence to describe large-amplitude magnetic waves in high-β plasmas. The high-β condition could be common in the case of the magnetosheath downstream of quasi-parallel shocks or behind the high Mach number bow shock. This would also likely be the case in solar dynamo and many astrophysical processes. The magnetosheath downstream of quasi-parallel shocks is of particular interest. Because only the tangential component of the magnetic field is amplified at the bow shock, the increase in the field strength is weak downstream of the quasi-parallel shock where the shock normal is nearly parallel to the magnetic field. The plasma beta (β) is often high there because of the compression and heating of the plasma at the shock while the field strength does not increase significantly. Large-amplitude fluctuations, (B/B0) ~ (βp), are one of the characteristics of this region. Figure 1 shows an example. More examples can be found in Engelbrecht et al. [1991] and Lin et al. [1991]. The important features to be noticed are: The fluctuations in the plasma thermal pressure are much smaller than that in the magnetic field and the fluctuations in each magnetic component make the value of the component both positive and negative. This latter feature leads to a small average of the component compared with the range of the value. Our theory applies only to the propagation region and does not discuss the regions where growth or damping is dominant. As we discuss in section 2, we study only quasi-stationary waves.

The mathematical description of nonlinear waves for linear polarization [Jeffrey and Taniuti, 1964; Korteweg and Petchek, 1966] and [Hollweg, 1971; Barnes and Hollweg, 1974; Lin and Lee, 1994] is in general different from that for circular polarization [Kendall et al., 1968; Huda, 1993]. For circular polarization, because of the 90° phase shift between the two transverse components of the perturbations, the nonlinear equation becomes either the Korteweg de Vries or the nonlinear Schrödinger equation. Our theory will describe linearly polarized fluctuations. Thus our theory applies to frequencies significantly below the ion gyrofrequency. Most previous nonlinear theories of linearly polarized waves are based on the method of characteristics. In this approach, the characteristics at each point of a wave propagate with a speed which is the same as the phase speed given by linear MHD theory and using the local values in order to calculate the speed. For a large-amplitude wave, the phase speed evaluated by the linear dispersion relation varies throughout a wave cycle. Therefore the shape of a compressional wave changes with time: Some portions compress and steepen and others rarefact, as illustrated in Figure 2a. Although this approach is very useful in the study of shocks, discontinuities and wave steepening, and in other theoretical investigations, observations from the magnetosheath have often shown little evidence for the steepening of some large-amplitude waves (see, for example, Figure 1). As we will see later in the paper, the primary wave activity in Figure 1 is of period 1.5-2 s which is well below the 5-s resolution of the data.

In a low-β plasma, steepening processes can be related to the nonlinearity of the thermal parameters because the thermal pressure is small and the perturbations can easily cause the nonlinear response in the thermal parameters. However, in a high-β plasma, the ion perturbation can reach a nonlinear stage much more easily than the thermal ones. Therefore nonlinearity does not necessarily lead to wave steepening. More discussion can be found in section 5.

Let us assume that a nonlinear magnetic perturbation has evolved to a quasi-stationary state as shown in Figure 2b, or the steepening rate or damping rate is much smaller than the frequency of the wave. The physical differences between the two wave patterns in Figure 2 are the following. For a steepening wave, the power in the higher harmonics increases as the wave front steepens. It becomes so important that the truncation at a certain harmonic in a Fourier expansion
approach is not valid. Waves at different frequencies and wavenumbers are strongly coupled. The power of a stationary wave, however, concentrates at the lowest harmonics with few wavenumbers and does not depend strongly on time. Different harmonics propagate at similar speeds and there is little dispersion among them. This permits the truncation at a low harmonic in a mathematical treatment. In fact, the small dispersion among harmonics effectively damps the nonlinear resonance in higher harmonics and hence prevents the waves from steepening. Further discussion will be provided in section 5. In the mathematical description given in the next section, we discuss only a single primary wave with its harmonics. In reality, several waves and modes could coexist at the same place and same time. We assume that they do not interact with each other.

In section 2 of this paper, we first derive the general equations for high-$B$ large-amplitude waves from MHD theory. In section 3, we then discuss the dispersion relations of the modes present. In section 4, we discuss the issues in data analysis of large amplitude wave and show an example. Discussion on several most important aspects concerning the approximations made in the procedure is given in section 5.

2. Perturbation Equations

The ideal MHD equations for an isotropic polytropic plasma are

\[
\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho + \rho \nabla \cdot \mathbf{v} = 0
\]  

\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}
\]  

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})
\]  

\[
\frac{\partial (\rho \mathbf{v})}{\partial t} + (\mathbf{v} \cdot \nabla) \rho \mathbf{v} = 0
\]

where $\rho$, $\mathbf{v}$, $p$, $\mathbf{B}$, $\gamma$, and $\mu_0$ are plasma density, velocity, thermal pressure, magnetic field, polytropic index, and the permeability in vacuum, respectively.
On the basis of observations, we make the following approximations: Perturbations in the magnetic field are of the order of one, 1/β (B/2m_0ρ)^1/2, and the order of the velocity component parallel to the propagation front is of order 1, the perturbed density and the normal component of the velocity are of order ε, and the temporal and spatial scales are of order 1 when they are scaled to the Alfven velocity.

The perturbations of (1)-(4) in the plasma frame, i.e., v_0 = 0, retaining terms of 2 leading orders are
\[ \frac{\partial \delta \rho}{\partial t} + (\delta v \cdot \nabla) \delta \rho + \nabla \cdot \delta \mathbf{v} = 0 \]  
(1')
\[ \frac{\partial \delta v}{\partial t} + (\delta v \cdot \nabla) \delta v = - \frac{\gamma \beta}{c^2} \nabla \rho + \nabla \cdot (\delta \mathbf{B} \cdot \mathbf{b}_0) + (\mathbf{b}_0 \cdot \nabla) \delta \mathbf{B} \]  
(4')

\[ \delta \mathbf{p} = \mathbf{q} \phi \delta \mathbf{B} \]  
(3')

where \( \rho, \mathbf{v}, \mathbf{b}_0, \) and \( \delta \mathbf{B} \) are perturbed quantities, and \( \mathbf{q}, \mathbf{p}, \mathbf{C}_A, \mathbf{b}_0, \) \( \mathbf{B}_0, \) and \( \mathbf{B}_0 \) are, respectively, the plasma \( \beta \) is evaluated using \( \mathbf{B}_0 \) and \( \rho_0 \). \( \partial / \partial t / \nabla \) is normalized by the Alfven velocity, subscript 0 denotes average quantities, and \( \mathbf{b}_0 \) is the unit vector of the average field. \( A (\delta \mathbf{B} \cdot \nabla) \delta \mathbf{B} \) term in equation (4') has been omitted because in one dimension the perturbed field is always perpendicular to the direction of the spatial derivative as required by the divergence free of the field. We point out that although we have stated a few approximations, to derive (1')-(4') requires only that the perturbed density is of order \( \epsilon \). Since the perturbed pressure is proportional to the perturbed density, we use the perturbed density, instead of the perturbed pressure, as the diagnostics in the following discussion. In a conventional approach, the perturbed quantities could be further expanded in orders; here we expand them in the Fourier space. The ordering for this expansion is that the tangential field and velocity perturbations of the fundamental mode and second harmonic are of order 1, the perturbations of all other quantities of the first two harmonics are of order \( \epsilon \), and all perturbations of the third harmonic and higher harmonics are at least another order smaller. The ordering used in this work is summarized in Table 1. The physical picture for our approach is the following: We introduce a large amplitude primary wave in the fundamental mode into the system to study the responses of the system. Because the plasma \( \beta \) is high, the plasma perturbations are small except the tangential velocity which is directly affected by the field perturbation. The nonlinear effects of the tangential field and velocity perturbations will generate harmonics in the system. If the nonlinear steepening rate is small, as is seen in observations and will be discussed in section 5, the power in higher harmonics drops rapidly. Let the perturbation of a parameter in the nth

<table>
<thead>
<tr>
<th>Table 1. Order of Quantity</th>
<th>Order</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>( \rho )</td>
<td>( \epsilon )</td>
<td>( \epsilon )</td>
<td>( \epsilon )</td>
</tr>
<tr>
<td>( \mathbf{b}_0 )</td>
<td>( \mathbf{b}_0 )</td>
<td>( \epsilon )</td>
<td>( \epsilon )</td>
<td>( \epsilon )</td>
</tr>
<tr>
<td>( \mathbf{v}_0 )</td>
<td>( \mathbf{v}_0 )</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>( \mathbf{v}_1 )</td>
<td>( \mathbf{v}_1 )</td>
<td>( \epsilon )</td>
<td>( \epsilon )</td>
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</tr>
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</table>

Subscripts \( i \) and \( s \) denote the components tangential and normal to the perturbation front, respectively.
Substituting (14) into (11) yields

\[ k \cdot v_2 = 0 \]  

(15)

Using (14) and (15), the remaining equations in the equation set (5)-(10) can be simplified as

\[ \frac{\gamma \beta}{2} p_1 + B_1 \cdot b_0 + \frac{1}{1} B_1 \cdot B_2 = 0 \]  

(16)

\[ \frac{\gamma \beta}{2} p_2 + B_2 \cdot b_0 + \frac{1}{1} B_2 \cdot B_1 = 0 \]  

(17)

k \cdot b_0 = 0  

(18)

For the nonpropagating mode, the velocity perturbations must be along the front of the structures and determine the stability and field perturbations. Equations (16) and (17) give the pressure and field conditions. The first harmonic density perturbation is out of phase with the field perturbation. Equation (16) indicates that the mode has the characteristic of a series of tangential discontinuities.

3.3. Purely Incompressible Mode

For the purely incompressible mode with \( p_1 = p_2 = 0 \), equations (5)-(10) become

\[ k \cdot v_1 = k \cdot v_2 = 0 \]  

(19)

\[ \omega v_1 = (B_1 \cdot b_0) - (k \cdot b_0) B_1 + \frac{1}{1} (B_1 \cdot B_2) k \]  

(20)

\[ \omega B_1 = -(k \cdot b_0) v_1 \]  

(21)

\[ \omega v_2 = (B_2 \cdot b_0) - (k \cdot b_0) k + \frac{1}{1} B_2 \cdot k \]  

(22)

\[ \omega B_2 = -(k \cdot b_0) v_2 \]  

(23)

Equations (20) and (21) require that \( B_1 \) and \( v_1 \) are parallel and both are perpendicular to \( k \). Similarly, \( B_2 \) is parallel to \( v_2 \), and all are perpendicular to \( k \). Note however that \( B_2 \) and \( v_2 \) are not necessarily parallel to \( B_1 \) and \( v_1 \).

Let \( x \) and \( y \) be the directions of \( B_1 \) and \( k \), respectively, and \( z \) be normal to the plane containing \( B_1 \) and \( k \) and complete the right-handed system, then break (20)-(23) into components, and retain terms up to the first order. This yields

\[ b_{0z} + \frac{1}{2} B_{2z} = 0 \]  

(24)

\[ v_{1z} + b_{0z} f_1 = 0 \]  

(25)

\[ v_{1z} + b_{0z} f_1 = 0 \]  

(26)

\[ b_{0z} B_{2z} + b_{0z} B_{2z} + \frac{1}{2} B_{2z} = 0 \]  

(27)

\[ v_{1z} + b_{0z} B_{2z} = 0 \]  

(28)

\[ v_{1z} + b_{0z} B_{2z} = 0 \]  

(29)

\[ v_{1z} + b_{0z} B_{2z} = 0 \]  

(30)
\[ v_{n_3 B_2} + b_0 v_{\phi_4} = 0 \]  

(31)

The phase velocity for both harmonics of the purely incompressible mode, \( v_\phi = \omega / k \), is

\[ \eta^2 = \cos^2 \theta \]  

(32)

where \( \theta = \cos^{-1} b_0 \) is the angle between the propagation direction and background field. This result is consistent with the common understanding that the Alfven mode or intermediate mode is nondispersive and its dispersion relation does not change even in nonlinear stages.

For completeness, one can verify that for waves propagating with the intermediates velocity, or (32), using the solutions derived from (6), (7), (9), and (10), the density perturbations are zero with the accuracies specified by (5) and (8).

In order to have a propagating mode, either \( b_0 \) or \( b_4 \) must not be zero, from (27), and \( v_\phi \neq 0 \), from (32); that is, the mode can only propagate obliquely to the field. The mode, however, does not require coplanarity, or \( b_0 \) and \( B_2 \), do not need to be zero and are

\[ b_0 = -\varepsilon_0 b_4 = \frac{B_{0z}}{4} \]  

(27')

When \( b_0 = -\varepsilon_0 b_4 \), \( B_2 \) can be zero, or \( B_2 \) is perpendicular to \( B_1 \) and \( k \), and \( B_4 \) is perpendicular to the plane containing \( b_0 \) and \( k \), the same as the linear Alfven mode. It is important to point out that the results do not depend on the plasma \( \beta \). This mode can exist in plasmas of any \( \beta \). In an analysis of the weakly nonlinear Alfven mode, Hohla [1971] found the generation of the second harmonic but found no purely incompressible mode in his system. In his analysis, he did not consider the possibility of noncoplanar second harmonic perturbations. His mode is consistent with our incompressible mode with \( r_1 = 0 \) and \( r_2 \neq 0 \), which will be discussed as a limiting case for compressible modes.

3.3. Compressible Mode

When \( r_1 \neq 0 \) and \( r_2 \neq 0 \), the combination of (5) and (8) yields

\[ \nu_1 + \frac{1}{2} (k \cdot v_2) |r_1| = (k \cdot v_1) \left( \frac{1 + \frac{1}{2} (k \cdot v_1)}{\omega} \right) \]  

(5')

\[ \omega_2 = (k \cdot v_1) + \frac{1}{4 \omega} (k \cdot v_1)^2 \]  

(8')

We eliminate \( p_1 \) and \( p_2 \) in (6) and (9) with (5') and (8') and obtain

\[ \nu_2 + \frac{1}{2} (k \cdot v_1) v_1 = \frac{\gamma B_2}{2 \omega} (2 \omega + (k \cdot v_2)) (k \cdot v_1) k \\
- \left( b_0 \cdot B_1 \right) B_1 + \left( b_2 \cdot B_1 \right) B_2 + \frac{1}{2} \left( B_1 \cdot B_2 \right) k \]  

(8')

\[ \omega_2 - \frac{1}{4} (k \cdot v_1) v_1 \left( \gamma B_2 \right) (k \cdot v_1)^2 \left( \frac{1}{4 \omega} (k \cdot v_1)^2 \right) k \\
- \left( b_0 \cdot B_1 \right) B_1 + \left( b_2 \cdot B_1 \right) B_2 + \frac{1}{2} \left( B_1 \cdot B_2 \right) k \]  

(9')

From (9') and (7), the magnetic field and velocity perturbations of the first harmonic are coplanar with the wavevector and the background field, i.e., \( B_2 = 0 \), but the perturbations of the second harmonic are not necessarily coplanar. We break (6'), (7), (9'), and (10) into components and retain terms up to the first order. This yields

\[ \left[ \nu_\phi - \gamma B_2 / 2 \nu_\phi \left( 1 + \frac{1}{2} (k \cdot v_2) \right) v_\phi B_1 \sin \theta \cos \theta B_1 = 0 \]  

(33)

\[ \left( \nu_\phi + \frac{1}{2} \nu_\phi \right) v_\phi B_1 \cos \theta B_1 = 0 \]  

(34)

\[ \left( \nu_\phi - \frac{1}{2} \nu_\phi \right) B_1 \sin \theta B_1 \cos \theta B_1 = 0 \]  

(35)

\[ \nu_\phi B_1 - \frac{1}{4} \nu_\phi B_1 \cos \theta B_1 = 0 \]  

(36)

\[ \left( \nu_\phi - \gamma B_2 / 2 \nu_\phi \right) v_\phi B_1 \sin \theta B_1 \cos \theta B_1 - \frac{1}{4} \nu_\phi B_1 \cos \theta B_1 = 0 \]  

(37)

\[ \nu_\phi B_1 \sin \theta B_1 \cos \theta B_1 - \frac{1}{4} \nu_\phi B_1 \cos \theta B_1 = 0 \]  

(38)

\[ \nu_\phi B_1 \sin \theta B_1 \cos \theta B_1 = 0 \]  

(39)

\[ \nu_\phi B_1 \sin \theta B_1 \cos \theta B_1 = 0 \]  

(40)

where \( \eta_\phi \) is the phase velocity of the compressible mode. The off-plane perturbations to the second harmonic have no effects on other components and are of no interest in our problem. The first-order correction in the phase velocity creates first-order perturbations in \( B_2 \) and \( v_\phi B_2 \) in (39) and (40). We will not further discuss this component other than noting that observationally wave power may appear in the noncoplanar component in the second harmonic.

The zeroth order dispersion relation can be obtained from the zeroth order terms in (34) and (35). It is

\[ \nu_\phi B_1 = \cos \theta \]  

(41)

Let \( \nu_\phi = \nu_\phi B_1 + \Delta \) where \( \Delta \) is the first-order correction. From (34) and (35), eliminating \( \nu_\phi B_2 \) obtains

\[ \nu_\phi B_1 = \frac{2 \Delta}{\sin \theta} B_1 \]  

(42)

Combining (35) and (42) yields

\[ \nu_\phi B_1 - \frac{1}{2 \Delta} (\nu_\phi B_2 + \frac{1}{2} \nu_\phi B_1) B_1 = 0 \]  

(43)

Similarly, other perturbations up to the first order are

\[ \nu_\phi B_1 - \frac{1}{4 \nu_\phi} (\nu_\phi B_2 - 4 \cos \theta B_2) \]  

(44)
The perturbations in the field strength, |B| and |B|, are important in data analysis and are

\[ |B| = |B_1| \sin \theta + \frac{1}{2} |B_2| \sin \theta \]  

(47)

\[ |B_2| = |B_1| \sin \theta + \frac{1}{2} |B_2| \sin \theta \]  

(48)

The zeroth order terms in (33) and (37) are

\[ \frac{\gamma B_1}{2\nu_{pl}} v_{pl} + B_1 \sin \theta + \frac{1}{2} |B_2| \sin \theta = 0 \]  

(33')

\[ \frac{\gamma B_1}{2\nu_{pl}} v_{pl} + \sin \theta B_1 \sin \theta + \frac{1}{2} |B_2| \sin \theta = 0 \]  

(37')

From (33'), (37'), (42), and (45), eliminating \( v_{pl}, B_2 \), and \( v_{pl} \) yields the first-order correction of the dispersion relation

\[ \left( \frac{\gamma B_1}{2\nu_{pl}} \right) \left( \sin^2 \theta + \frac{3}{16} |B_2| \sin^2 \theta \right) \left( 2 \sin^2 \theta + \frac{3}{8} |B_2| \sin^2 \theta \right) + \left( \sin^2 \theta - \frac{1}{8} |B_2| \sin^2 \theta \right) n^2 = 0 \]  

(49)

The solution of the first order correction due to the nonlinear effects is

\[ \Delta \nu_{pol} = - \frac{1}{\gamma \beta} \left( \sin^2 \theta + \frac{3}{16} |B_2| \right) B_1 / B_1 \sqrt{9|B_2| + 128 \sin^2 \theta} \]  

(50)

When \( \theta \) is zero, the dispersion relation corresponding to the plus sign in equation (50) gives

\[ \Delta \nu_{pol} = \frac{3}{8\sqrt{3}} |B_2| \]  

(51)

The solution given by the above expression to (42) or (45), for example, becomes unphysical. Therefore only the minus sign in equation (50) is valid, namely,

\[ \Delta \nu_{pol} = - \frac{1}{\gamma \beta} \left( \sin^2 \theta + \frac{3}{16} |B_2| \right) B_1 / B_1 \sqrt{9|B_2| + 128 \sin^2 \theta} \]  

(51')

For the nearly parallel propagating mode, when \( \sin \theta \ll 9|B_2|/128 \), the dispersion correction becomes

\[ \Delta \nu_{pol} = \frac{1}{3|B_2|} \sin^2 \theta \]  

(52)

The amplitudes of the fluctuations in the density and field strength are

\[ \rho_1 = \frac{1}{T \sqrt{\pi}} \sin \theta |B_1| \]  

(53)

\[ \rho_2 = - \left( \frac{1}{2|B_2|} \right) |B_2| \]  

(54)

\[ \frac{\left| B_2 \right|}{\left| B_1 \right|} = - \frac{1}{2|B_2|} \sin \theta \]  

(55)

\[ \frac{\left| B_2 \right|}{\left| B_1 \right|} = \frac{1}{2} |B_2| \sin^2 \theta \]  

(55)

For \( \theta = 0 \) we have \( \Delta \nu_{pol} = \nu_{pol} = v_{pl} = B_2 = 0, \rho_1 \neq 0 \), and \( \rho_2 \neq 0 \). Namely, for parallel propagation, the phase velocity is the Alfvén velocity, and the first harmonic and field strength perturbations are zero but the second harmonic density and field strength perturbations are not zero. Although the perturbation in the second harmonic field components may be of first-order (although not necessary, because of the possible zeroth-order perturbation in the \( x \) component), the first harmonic field perturbation generates zeroth-order perturbations in the second harmonic field strength, which then in turn drives pressure and density perturbations of the second harmonic.

When \( |B_2| < 128 \sin^2 \theta /\beta \),

\[ \Delta \nu_{pol} = - \frac{1}{\gamma \beta} |B_2| \sin^2 \theta \]  

(56)

4. Data Analysis

One of the most important issues is in treating large amplitude field fluctuations in \( < |B| > \neq | < B| > \), where \( < > \) denotes averaging in time domain. The difference is significant in the evaluation of the Alfvén velocity, plasma \( \beta \), and relative wave amplitude of the field. Given a field perturbation \( B = (B_0 + B_1 \cos \omega t, B_0, 0) \), the average field is \( < B > = B_0 \), and the magnitude of the average field is

\[ | < B > | = \sqrt{B_0^2 + B_1^2} \]  

(57)

The magnitude of the field is

\[ B = |B| = \sqrt{(B_0 + B_1 \cos \omega t)^2 + B_0^2} \]  

The average field strength is

\[ | < B > | = \frac{1}{T} \int_{-\tau}^{\tau} |B| dt \]  

(58)

The difference between \( < |B| > \) and \( | < B| > \) depends on the propagation angle. In general, \( | < B| > \) is greater than \( < |B| > \). They become the same for perpendicular propagation when \( B_0 > B_1 \). The Alfvén velocity defined in this work corresponds to that calculated using \( < |B| > \).

A most important phenomenon when analyzing large amplitude field fluctuations is the appearance of the higher harmonics in the field strength as illustrated in Figure 2 and discussed in sections 2 and 3. The amplitudes of the harmonics in the field strength are
\[ |B_{\text{m}}| = \frac{2}{T} \int_{T}^{T+\frac{\pi}{2}} B \cos \omega t \, dt \quad m = 1, 2, 3, \ldots \] (59)

There is no general analytical expression for the integral in (59). The first harmonic in the field strength would be dominant if \( B_{20} > B_1 \). The second harmonic would be important if \( B_{20} < B_1 \). Therefore one may expect a significant presence of the second harmonic in the field strength for parallel propagation but not for perpendicular propagation. Figure 3a shows the Fourier spectra of the field strength for waves with different angles between \( B_0 \) and \( B_1 \) when \( B_0 = B_1 = 1 \). It is obvious that the power in the first harmonic increases as the angle decreases and that the power in the second harmonic increases with the angle. As can be seen in Figure 3a and will be further discussed in section 5, higher harmonics are unimportant in this system. The second harmonic in the field strength is a major indicator of the nonlinear, electron effects and creates problems in using wave analysis schemes based on linear theory (e.g., Song et al., 1994). The problem arises because the second harmonic in the field strength does not correspond to variations in the components. The effects are most severe for parallel propagation.

The observational characteristics of the modes discussed in section 3 are the following. For the nonpropagating mode, in addition to the oscillations in the field and velocity components, power wave will appear in both thermal pressure (and density) and the magnetic field strength (and magnetic pressure) in both the first and second harmonics. However, the wave power in the total pressure, the sum of the thermal and magnetic pressures, should be very weak. For the purely incompressible mode, oscillations should be present in the field and velocity components but small in the density and field strength in both harmonics. For the compressible modes, in general, large wave power should be seen in the field and velocity components for the first harmonic. The wave power in the second harmonic of the field and velocity components depends on the propagation angle. It is large for quasi-perpendicular propagation and is small for quasi-parallel propagation. The perturbations in the density and field strength are significant (the first order) for the second harmonic but may not be significant for the first harmonic for quasi-parallel propagation.

In order to use the results from our simple stationary system to explain a complicated observed wave pattern, one may assume that there exist a variety of wave modes, each of which has a particular propagation direction at a particular frequency but the interaction among waves is weak, and that a single wave is dominant at a

![Figure 3](image-url)

**Figure 3.** (a) Fourier spectra of magnetic field strength of stationary waves with \( B_0 = |B_1| = 1 \) and \( f_1 = 0.001 \text{ Hz} \), or \( \Omega = (\sin 2\pi f_1 t + \cos \phi \sin \phi, 0) \) where \( \phi \) is the angle between \( B_0 \) and \( B_1 \). When \( \phi = 0^\circ \) (thick solid line), the wave power is concentrated on the fundamental mode. When it is \( 90^\circ \) (thin solid line), there is no power at odd harmonics and the second harmonic carries most power. For other cases, the energy couples to high harmonics but with a rapid decrease in the power. The spectrum for the primary wave \( B_1 \) is the same as the thick solid line. (b) Fourier spectrum of a steepened wave or sawtooth-shape fluctuations with \( B_0 = |B_1| = 1 \), or \( B = \cos^{-1}(\tan \beta f_1 t/\pi + 1) \). The wave power is distributed in all harmonics. In comparison with Figure 3a, the power in higher harmonics is much greater for sawtooth-shape fluctuations.
particular frequency. As a spacecraft changes its location in the magnetosphere, due to the motions of the spacecraft, the magnetopause, and the bow shock, it is sampling different combinations of these wave modes. It is worth mentioning that the quasi-stationary wave assumption is required only for the time interval during which a spectrum is obtained and not necessarily for the entire magnetosheath; that is, although a wave may change through the magnetosheath, it may be considered as being stationary in each small region within the sheath.

We note that in the magnetosphere, a class of ultra-low frequency (ULF) pulsations shows double frequency oscillations in the field strength [e.g., Cummings et al., 1969]. From a purely observational viewpoint, these double frequency ULF oscillations are seen in the compressional component, while the wave we are modeling does not need to have a compressional component. From a theoretical point of view, the wave we are modeling is a propagation mode in free space while the frequency doubling in the magnetosphere occurs in a confined space associated with resonances [e.g., Southwood and Kivelson, 1997], although both are due to nonlinear effects.

For the case shown in Figure 1, the field strength defined by (57) is about 7.2 nT and that defined by (58) is 14.7 nT. Accordingly, the plasma β is about 34 and 8, respectively. The amplitude of the waves is about 8 nT. Figure 4 shows the spectra of these waves. The scale for each of the upper, middle, and lower waves has been shifted by a decade upward. The dashed lines indicate spectra of pink noise; the energy density of which is independent of frequency. The level of the noise in components is determined according to the noise level of the component with the maximum noise, or the Bz, in the lowest frequencies. The most significant peak at the low-frequency end of each component, as marked as F1, F2, and F3, has no significant corresponding peak in the spectrum of the field strength. This is consistent with nearly parallel propagation large amplitude waves as shown by the thin solid line in Figure 3a. As expected by the theory, peaks in the spectrum of the field strength clearly appear at the frequencies twice the frequencies at the peaks in the components, as marked as 2F1, 2F2, and 2F3. At least one of them, 2F2, has no corresponding peak in the spectra of the components, which is again consistent with the parallel propagation compressible mode. These observational features lend strong support to our theoretical approximations and treatment. Furthermore, the peaks in the field strength cannot be produced by the purely incompressible mode. The compressibility of the wave makes it possible to transport energy across the magnetopause.

Observations are made in the spacecraft frame and our theory is derived in the plasma frame. As long as the average velocity does not vary strongly in space for example, |Δv∥| < |Δv⊥|, our theory is applicable, but the frequency should be Doppler shifted by k·v∥.

Figure 4. Fourier spectra of the magnetic field for the case shown in Figure 1. The scales for the three components have been shifted upward by one decade cooperatively. The dashed lines indicate spectra of a slope of -1. The most significant peak at the low frequency end of each component, as marked as F1, F2, and F3, has no corresponding peak in the spectrum of the field strength, but there are peaks at the frequencies, twice those in the field strength, as marked as 2F1, 2F2, and 2F3.

Unfortunately, the direction and the magnitude of the wave vector is very difficult to determine in observation unless simultaneous measurements from several points are available. The Doppler shift is generally unknown. An interesting observational feature is that the measured plasma density fluctuations are often not small during quasi-parallel sheath passes. The temperature
varies in anti-phase with the density, which makes the pressure variations weak as we assumed in our theory. Although this anti-correlation can be physical, we think it may be instrumental. Song et al. [1997] showed that because of the lower cutoff energy of the particle detector and spacecraft charge, the density and the temperature can vary in anti-phase when the velocity changes. The amplitude of the artificial density and temperature fluctuations can be large when the velocity fluctuates strongly.

5. Discussion

5.1. Convergence Rate at High Harmonics

The theory described in section 2 is based on the approximation that the two lowest harmonics carry most of the wave energy so that the perturbations can be effectively truncated at the second harmonic. In comparison, the underlying assumption of conventional nonlinear approaches is that higher harmonics are always important although each of them may not seem large. Therefore the question becomes whether the base function that we are using converges fast enough to describe the wave pattern with which we are dealing, so that our second harmonic truncation and the ordering shown in Table 1 do not neglect anything important. From Figure 2b, we know that the parameter which carries most information of nonlinear effects is the field strength. In the nonlinear stage, a perturbation in either the field direction or magnitude will result in a change in the field strength. Then the perturbations in the field strength induce plasma perturbations. The power of different harmonics of the wave pattern investigated in this paper, Figure 2b, is shown in Figure 3a. The power in the third harmonic is more than 2 orders less than the primary wave, translating to 1 order in amplitude. The power in higher harmonics drops rapidly. The sum of the powers of high harmonics should converge quickly. In comparison, Figure 3b shows the spectrum of a steepened wave pattern, or a sawtooth shape wave as seen in Figure 2a. It is obvious that the power in higher harmonics drops very slowly. An analytical expression shows that the Fourier parameter drops as 1/\(m^n\). The sum of the energy of high harmonics does not converge. This comparison may justify our approach and ordering.

5.2. Detuning

Under our ordering scheme, the approximations and the results are completely self-consistent only for the nonpropagating mode, and for the purely incompressible mode when \(R_{\lambda} = 8\). In other words, for these modes, the amplitude in the third and higher harmonic due to the beating effects of the first two harmonics is indeed of or less than second order, and the beating effects of higher harmonics transfer little energy to the lower harmonics. For other modes, the nonlinear effects will drive the third harmonic to a significant amplitude. In order to justify our truncation for complete self-consistency, other mechanisms are needed to detune or damp the resonance by the first two harmonics. We recall that when ion cyclotron effects are included, the phase velocity is reduced from the Alfvén velocity as the frequency approaches the ion cyclotron frequency. Therefore there is an increasing dispersion for higher and higher harmonics. This difference in the phase velocity will detune the higher harmonics from resonating with the lowest ones. The detailed analysis of such detuning processes will be reported elsewhere. We noted that the frequency of the third harmonic in our example is close to the ion cyclotron frequency. The ion cyclotron frequency is 0.1 Hz, and 3F\(_1\), 3F\(_2\), and 3F\(_3\) are 0.06, 0.075, and 0.12 Hz, respectively. If we assume that the Doppler shift is unimportant, these values indicate that the cyclotron resonance is the most likely mechanism that detunes the beating resonance.

5.3. Sonic Modes

We have investigated various possible orderings when 1/\(\beta_0\) is of order \(e\) and 8\(\beta\) is of order 1. We have not been able to find an ordering that supports a phase velocity of the order of the sound speed when the pressure fluctuations are small or when the wave is not strongly steepened. This means that it is very unlikely for a sound wave in a high-\(\beta\) plasma to produce a large magnetic field perturbation while keeping the pressure variations small, as shown in Figure 1. For waves with a phase velocity of the order of sound speed, from (3), the perturbed velocity scaled to the sound speed is of the order of the perturbed field. When the velocity is scaled to the sound speed, factor \(\gamma/\beta/2\) in the first term on the right-hand side of \(\gamma/2\) \(\gamma/2\) needs to be added to the terms of the magnetic field. Therefore the effects of the magnetic forces are reduced by one order. When the perturbed field is of order 1, the perturbed velocity is of order 1 according to (3) and then the perturbed density has to be of order 1 in order to balance the left-hand side terms in (2). The consequence of the large density perturbation is the strong steepening of the wave as we see in the case of fast shocks. Our finding is significant because, contrary to common belief, in a high-\(\beta\) plasma large-amplitude compressional waves cannot be sound waves if they are not strongly steepened. It is not the magnetosonic wave but the Alfvén and slow waves that carry most magnetic wave energy if the amplitude of the wave is large. The magnetosonic mode is unlikely to produce significant field perturbations without significantly steepening the wave.

5.4. Steepening Rate

Based on observations, we have assumed in our theory that the wave steepening is weak, or the wave has the
pattern shown in Figure 2b and not in Figure 2a. The
steeping time can be estimated by
\[ \tau = \frac{\lambda}{\delta (v_\perp + v_{\parallel pl})} \]
where \( \lambda \) is the wavelength, \( v_\perp \) is the plasma velocity
along the propagation direction and \( \delta (v_\perp + v_{\parallel pl}) \) is the ve-
locity difference between two sequential fluid elements.
When \( \delta (v_\perp + v_{\parallel pl}) \) is positive the trailing element is
able to catch up to the leading one and then the wave steepens.
From linear theory [Kraus of and Pochelok, 1966],
\[ \frac{\delta (v_\perp + v_{\parallel pl})}{v_{\perp pl}} \frac{\delta \rho}{\delta \rho} = \frac{3}{2} \left( 2 \gamma \cos^2 \theta - \frac{v_{\perp pl}^2}{v_{\perp pl}^2} \right) \]
For high-beta plasmas, the value of the right-hand side is
between 1 and 1.5. Thus the steepening rate of the wave is
controlled by the density perturbation and is
\[ \gamma_c = 1/\tau = \frac{\delta (v_\perp + v_{\parallel pl})}{\lambda} \leq \frac{3}{2} \left( 2 \gamma \cos^2 \theta - \frac{v_{\perp pl}^2}{v_{\perp pl}^2} \right) \]
As the pressure fluctuations are weak, say \( \delta P \approx 0.2 \),
the steepening rate is much smaller than the frequency;
namely, the waves will not significantly steepen in a
wave period. The physical reason for this result is that the
steeping process relies on the nonlinearity in the plasma
parameters, the perturbations of which are lin-
ear in our problem. Therefore we conclude that the
steepening of the waves is relatively unimportant for
the waves we are studying.
6. Summary
We have developed an ordering and nonlinear theory of
MHD waves in a high-beta plasma based on observa-
tional evidence. We assume that the wave is quasi-
stationary and does not couple with other waves. We
consider the reciprocal plasma beta and the pressure
perturbation as small parameters, while the magnetic
field perturbation is not small. Retaining nonlinear
terms associated with the field and velocity perturba-
tions, we expand the perturbations in the frequency
mainly assuming that the power is higher harmonics falls
rapidly. The approximation effectively truncates the
problem at the second harmonic. There is a nonprop-
agating mode in the system which is similar to a series of
tangential discontinuities. A purely incompressible
mode exists in the system. There is a compressional
mode in the system. The nonlinear compressional mode
is similar to the linear mode with a nonlinear modi-
fication. A most important feature of the compressional
mode is that there is significant wave power in the sec-
ond harmonic; in the density and field strength spectra
while the power in the fundamental mode in these spec-
tra could be very small. By examining possible order-
ing and perturbation relations, the modes correspond-
ing to the fast mode cannot support large field pertur-
bations without strongly steepening the waves. This in
fact poses a serious question as to the significance of un-
steepened fast mode waves, especially in causing large
magnetic perturbations, in a high-beta plasma. We note
that although the magnetosonic modes are commonly
believed to be important in high-beta plasmas they have
not been significantly observed in the magnetosphere.
The only large-amplitude fast wave observed is the bow
shock which is strongly steepened and not periodic.
In data analysis, to evaluate the Alfvén velocity, plasma
beta, and normalized field perturbation, one should use
the magnitude of the average field instead of aver-
age of the field strength. Because the nonlinear effects
couple a wave to its second harmonic, extra caution
should be taken when analyzing large amplitude mag-
netic fluctuations. For parallel propagating waves, it is
possible that the primary wave does not show in the
effect of the field strength at its frequency but at its
doubled frequency. Kinetic effects which have not been
included in our theory may result in circular polariza-
tion for which our Fourier expansion does not apply and
may provide the detuning mechanism to prevent high
harmonics from resonantizing. Other kinetic effects such
as Landau damping and phase speed reversal in the
high-beta plasmas are not included. Nonlinear kinetic
effects will result in particle trapping which is also not
included in the model. We think it important to verify
our theory by numerical simulations.
Although the magnetosphere is filled with waves, in order
for these waves to be transmitted into the inter-
planetary medium, it must be compressible propagat-
ing modes because in the absence of active resonant
processes the magnetopause behaves like a tangential discontinui-
ity most of the time through which the oscillations in the
total pressure are a most effective mechanism to tran-
smit the wave energy. Our theory shows the possibility of
such a mode downstream of the quasi-parallel shock,
and our data analysis indicates its possible existence.
Therefore the compressional mode shown in this work
provides one of the most likely mechanisms to explain
the enhanced wave activity observed in the magnetosphere,
response of the quasi-parallel shock.
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